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Generalizing 2D Geometric Properties to 3D With the Aid of DGS

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Abstract

In this paper, we would show how we could use computer programs in geometry researches. This paper contains the processes of generalizing the properties of 2D-geometry to that of 3D-geometry. In addition, we would show how we could generalize backward; we could generalize further on 2D-geometry using the properties we found in 3D-geometry. In our research, we tried to generalize triangular properties into polygons and tetrahedrons. The computer tools we used are GSP and Cabri 3D. We used GSP for 2-dimensional researches and Cabri 3D for 3-dimensional researches.

1. Introduction

Everyone can study Geometry whenever he has some things to draw with and on, such as a sketch board. However, in this classical way of research, using these boards, one cannot assure the accuracy of his research. That is mainly because it is hard to draw a figure exactly. What is worse, it is even harder for 3-Dimensional figures. With a development of computer technology, drawing and visualizing these figures became much easier. This led the researchers to enhanced accuracy and efficiency in geometry studies.

In this paper, we used GSP to study 2D Geometry and Cabri-3D to study 3D Geometry as DGS software. GSP is one of the most renowned computer tools around the world in studying 2D Geometry. However, Cabri-3D is not the one used frequently because it is not yet familiar to people around the world.

This paper focused on giving examples of applying computer tools on geometry researches. Several topics we introduced were ones, which are really familiar and elementary. In chapter 2, we showed how we could generalize triangle centers to tetrahedrons. We described about generalizations of Menelaus’ theorem to polygons and polyhedrons, and about backward generalization, from polyhedrons to polygons in chapter 3. In chapter 4, we showed how we could find a property using dynamic methods of visualizing and performing experiments. We used a method of dynamic modeling using computer tools. We could precede our research very efficiently using computers. We performed our research following three steps:

First, using GSP, we studied about the two-dimensional properties in detail. Then, we inferred how it works on 2-dimensional planes. Also, we fully understood through performing a number of experiments using computer tools.

Second, we tried to generalize 2D properties to 3D space. In this process, we used Cabri 3D. We could fully visualize and try a number of figures. It was very helpful for performing a research on them. Compared to the non-dynamic methods, it was much easier to study, analyze and visualize the figures. We hypothesized our results from the experiments.

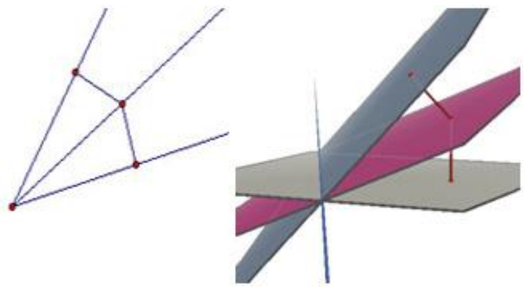
Third, through proving those found properties, we could check our conjectures we made before and then, we could explore the possibility of generalizing further. In addition, we could generalize backward.

With the aid of DGS and dynamic modeling method, the research proceedings become more efficient. Performing a number of experiments, we could easily figure out how we can generalize the properties. With the ideas from experiments in 2D, we could apply those ideas easily on 3D. This really helped our flow of thoughts. Generalizing triangle centers, especially for orthocenters, we could perform a bulk of trials and found proper conditions they exist. Also in chapter 3, since Menelaus' theorem is based on measurements of the line segments, we could calculate the results very effectively. If we had not those computer tools, we were not able to get results, because we have crucial difficulties on measuring distances on 3D. In addition, Fermat point in chapter 4 shows the method of trial/mistake approach. This is one of the best advantages of a research with computer tools.

2. Triangle Centers

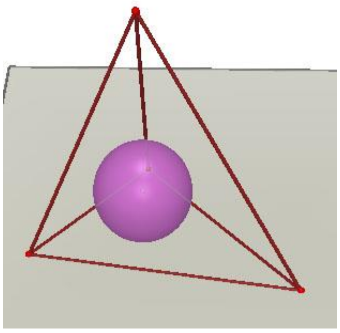
1. Incenters, Excenters, Centroids and Circumcenters

The definition of the incenter of a triangle is a center of triangle's inscribing circle, and be constructed as a point where three angle bisectors intersect. For any points on the angle bisector is equidistant to the two sides. Therefore, the point where three angle bisectors intersect is equidistant to the three sides. This means that it can be the center of the incircle.



[Figure 2 - 1] An angle bisector and an interior-dihedral angle bisector plane

We directly extended this concept into 3D space. If we construct interior-dihedral angle (=inner angle made by planar sides of tetrahedron) bisector plane, this plane consists of points which has same distance from two planar sides (of tetrahedron). There, we can find 6 interior-dihedral angles and drew six bisector planes for each angle. We deduced that the point would be equidistant to all facets of the tetrahedron if these six bisector planes intersect at a single point. Furthermore, this point would be the center of inscribing sphere, or insphere. We drew some figures and found that these six planes intersect at a single point and it is the center of the insphere.



[Figure 2 - 2] The incenter of a tetrahedron

Next, we proved that these 6 bisectors meet on a single point. Then we also proved that it is center of insphere. Not only for incenter, but also for circumcenter, centroid, and excenter, we had similar procedure. These 4 centers were easily generalized. For any tetrahedron, those 4 centers exist, and we succeeded to prove them. [Table 2-1] is indicating where those other 3 centers are and their properties.

[Table 2 - 1] Other centers of tetrahedron

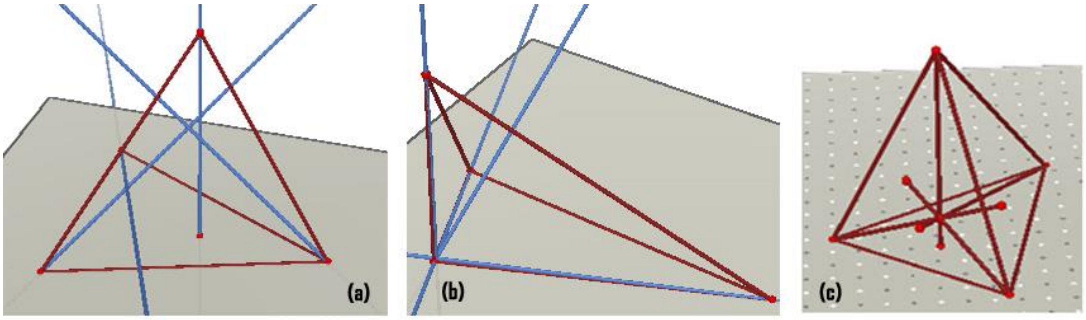
Centers	Position in 2D Geometry	Position in 3D Geometry	Property
Circumcenter	A point where three perpendicular bisectors intersect	A point where perpendicular bisecting planes intersect	Becomes the center of the circumcircle and the circumsphere, respectively
Centroid	A point where three medians intersect	A point where median planes (Planes with a edge and its opposite edge's middle point) intersect	Divides the line which connects a point and the opposite planes' centroid as 2:1, 3:1 respectively
Excenter	A point where exterior angle bisectors intersect	A point where exterior dihedral-bisecting planes intersect	Becomes the center of the excircle and the exosphere, respectively

A sample of those five centers is performed in this file: (Presented with Cabri3D, [1]).

2. Orthocenters

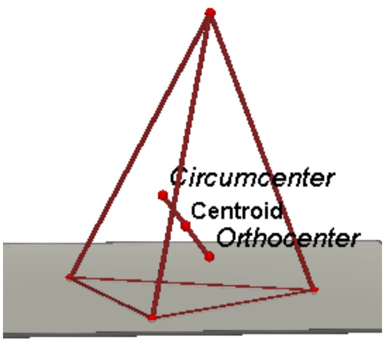
The orthocenter of a triangle is a point where three altitudes intersect. Then, this time, we proceeded to the generalization directly.

We generalized directly to 3D, and we defined the orthocenter as the point where the four altitudes of a tetrahedron intersect. We drew several figures. However, we soon found that the four altitudes for general tetrahedrons do not intersect each other. Then we started from a regular tetrahedron. For a right tetrahedron, the orthocenter existed. Using a dynamic modeling method, we checked if it could be generalized. (The orthocenter of a right tetrahedron, presented with Cabri 3D, [2])



[Figure 2 - 3] (a) General tetrahedrons (b) A counter example of the conjecture (c) Orthocenter of a tetrahedron

First, we moved a point vertically up and down, and the center existed. (This work is presented with Cabri3D, [3]) Then, we made a conjecture that the orthocenter exists when at least one facet is an equilateral triangle. However we found some counter examples, as shown in [Figure 2-3(b)]. (Presented with Cabri3D, [4]) We precede our research to generalize this condition of equilateral triangle. Finally, through a number of experiments, we found the orthocenter of a tetrahedron exists when an altitude intersects with opposite plane's orthocenter. With a set of experiments, we checked that the orthocenter exists under this condition. (Presented with Cabri3D, [5])



[Figure 2 - 4] Generalization of Euler's line

We proved that the four altitudes are concurrent on a point if at least one altitude intersects with the opposite plane's orthocenter. Also, we proved that the four altitudes intersect with their opposite facet's orthocenter if the orthocenter exists.

In addition, we've also shown that all orthotetrahedrons (=tetrahedrons with orthocenter) have circumcenter, centroid, and orthocenter to be collinear, as additional generalization of Euler's line [Figure 2-4]. (Presented with Cabri3D, [6])

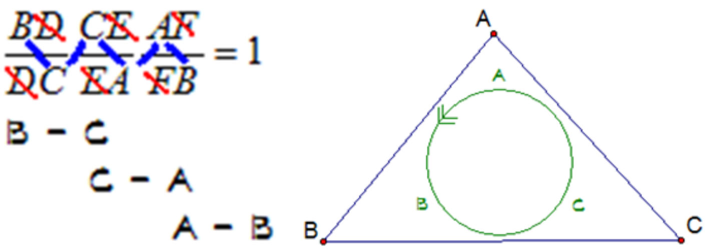
3. Menelaus' Theorem

Menelaus, a famous Egyptian mathematician, discovered the following theorem. This is what we will discuss in this whole chapter, and discuss about so-called "backward generalization".

Theorem(Menelaus) *Given triangle ABC and a line l that never passes neither A nor B nor C, let D, E, F as intersection point between line l and line BC, CA, AB, respectively. Then,*

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB}=1$$

The proof is introduced in [Kay01], pp.320 and shall be omitted. To generalize it into polygon, we observed the equation, $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB}=1$, and found that:

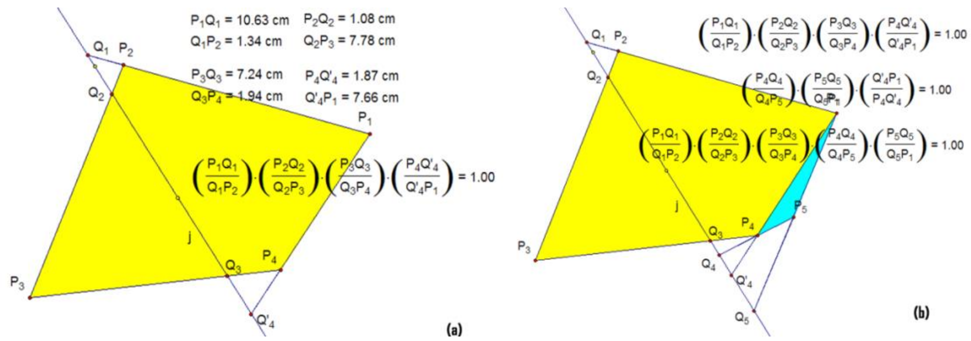


[Figure 3 - 1] Observing the pattern

That is, the vertices of triangle on the equation forms a "cycle"; namely, B-C-A. Keeping track of these points on triangle, we can see that it forms a cyclic curve. Also, between the points on cycle, we see that intersection point is placed on it (eg. Between B and C, there's D; hence corresponding ratio is BD/DC).

1. Generalization to Polygons

By the observation done at the prior, we could conjecture that if we take circular track of polygon to multiply ratios of internal division by a fixed line, then we can get 1 as we did on triangle. To check it so, we've drawn [Figure 3-2]. To be specific, we state this theorem as following:



[Figure 3 - 2] Observation for generalization

At left side of [Figure 3-2], we noted that a an equation holds for quadrilaterals, namely[1]

$$\frac{P_1Q_1}{Q_1P_2}\frac{P_2Q_2}{Q_2P_3}\frac{P_3Q_3}{Q_3P_4}\frac{P_4Q_4'}{Q_4'P_1}=1$$

To extend it so, we added auxiliary triangle, one side “stuck” on former quadrilateral – see the right side of [Figure 3-2]. In this diagram, we observed that:

$$\frac{Q_4'P_1}{P_4Q_4'}\frac{Q_4P_4}{P_5Q_4}\frac{Q_5P_5}{P_1Q_5}=1 \Rightarrow \frac{P_1Q_1}{Q_1P_2}\frac{P_2Q_2}{Q_2P_3}\frac{P_3Q_3}{Q_3P_4}\frac{P_4Q_4}{Q_4P_5}\frac{P_5Q_5}{Q_5P_1}=1$$

It is a similar equation that is held in (general) pentagon. Employing this “recursive” or “inductive” idea, we found a general rule that holds for a n-gon. That is, given n-gon $P_1P_2\ldots P_n$, and fixed line l (that never passes P_i), letting Q_i be intersecting point between line l and line P_iP_{i+1}, for i=1, 2, …, n (define P_{n+1} as P_1.), then we have:

$$\frac{P_1Q_1}{Q_1P_2}\frac{P_2Q_2}{Q_2P_3}\cdots\frac{P_{n-1}Q_{n-1}}{Q_{n-1}P_n}\frac{P_nQ_n}{Q_nP_1}=1$$

This equation is employed as our generalization of Menelaus’ Theorem at polygons.

2. Generalization to 3D

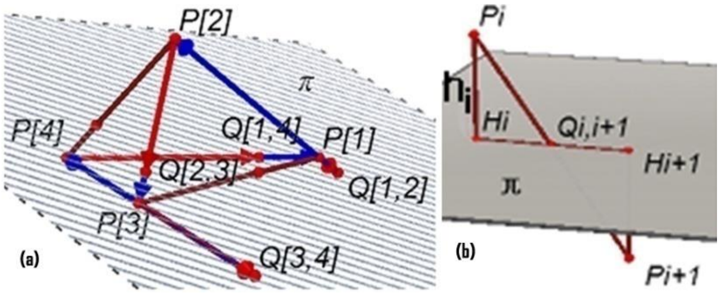
The idea of "cycle" takes central role in discovery of tetrahedral Menelaus' theorem. Though, Euler's contribution on graph theory tells us that we can't just loop through all edges for only once. Thus we can conclude that a full circular track can't be constructed on tetrahedrons. Though, when we just kept the idea of "closed loop" ONLY for tetrahedral generalization, we could obtain a result.

[1] In [Figure 3-2], P_1, P_2, P_3, P_4 all denotes vertices of yellow(big) quadrilateral. There’s line j passing through quadrilateral $P_1P_2P_3P_4$, and Q_1, Q_2, Q_3, Q_4` is intersection of line j with P_1P_2, P_2P_3, P_3P_4, P_4P_1 respectively. P5 is auxiliary point added and Q_5, Q_4 denotes intersection of line j with P_5P_1, P_4P_5, respectively.

Suppose, for a tetrahedron ABCD, let $P_1P_2\ldots P_nP_{n+1}$ be closed loop on that tetrahedron (so that $P_{n+1}=P_1$, P_i's are either A, B, C, or D, and $P_i \neq P_{i+1}$). Given plane π that never passes A, B, C, nor D, let $Q_{i,i+1}$ be intersecting point between π and line P_iP_{i+1}.

Further, let h_i is height from P_i to π. Accompanying [Figure 3-3] (b), we can see that triangle $P_iH_iQ_{i,i+1}$ and $P_{i+1}H_{i+1}Q_{i,i+1}$ is similar and $P_iH_i = h_i$, $P_{i+1}H_{i+1} = h_{i+1}$ and, $P_iQ_{i,i+1}/Q_{i,i+1}P_{i+1} = h_i/h_{i+1}$ in conclusion. If we multiply this equation through i=1 to n, we obtain:

$$\frac{P_1Q_{1,2}}{Q_{1,2}P_2}\frac{P_2Q_{2,3}}{Q_{2,3}P_3}\cdots\frac{P_{n-1}Q_{n-1,n}}{Q_{n-1,n}P_n}\frac{P_nQ_{n,1}}{Q_{n,1}P_1}=1$$



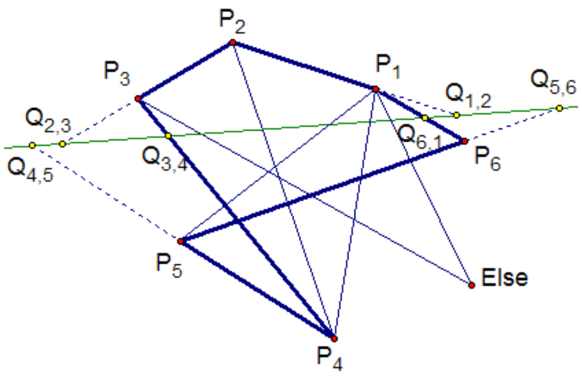
[Figure 3 - 3] (a) Menelaus' theorem for tetrahedrons (b) Height diagram

(Presented with Cabri3D: Multiple-loop on tetrahedrons, [7])

(Presented with Cabri3D: Circuit-concept generalization on 3D, [8])

This equation is what we will now say Menelaus' theorem for tetrahedrons. Although nowadays one can check this numerically with Cabri 3D but when we tried to observe this fact, Cabri 3D didn't provided numerical calculation and therefore all we could do was finding an approach to proof.

3. Backward Generalization



$$\left(\frac{P_1Q_{1,2}}{Q_{1,2}P_2}\right)\cdot\left(\frac{P_2Q_{2,3}}{Q_{2,3}P_3}\right)\cdot\left(\frac{P_3Q_{3,4}}{Q_{3,4}P_4}\right)\cdot\left(\frac{P_4Q_{4,5}}{Q_{4,5}P_5}\right)\cdot\left(\frac{P_5Q_{5,6}}{Q_{5,6}P_6}\right)\cdot\left(\frac{P_6Q_{6,1}}{Q_{6,1}P_1}\right) = 1.00$$

[Figure 3 - 4] Result of Backward-generalization

Note that Theorem 3.3 has alike equation but the points stated are not as the same situation as the points stated in Theorem 3.2. In Theorem 3.2, the point P_i's are aligned all vertices of polygon, but in Theorem 3.3, the point P_i's are aligned not all vertices on tetrahedron and therefore can be redundant. Though, cancellation of height is achieved by similarity of right triangles and therefore can be achieved in planar geometry and planar loop of vertices as well. This achieved our backward-generalization, and stated as following [Figure 3-4]:

Think of a graph (diagram with points and segments, on Euclidean metric) K on a plane and a line ℓ (on the same plane) such that it never passes points of K. Think of cycle (=path that has same starting point and endpoint) $P_1P_2\ldots P_nP_{n+1}$ on K, and suppose line P_iP_{i+1} meets with ℓ at $Q_{i,i+1}$. Then,

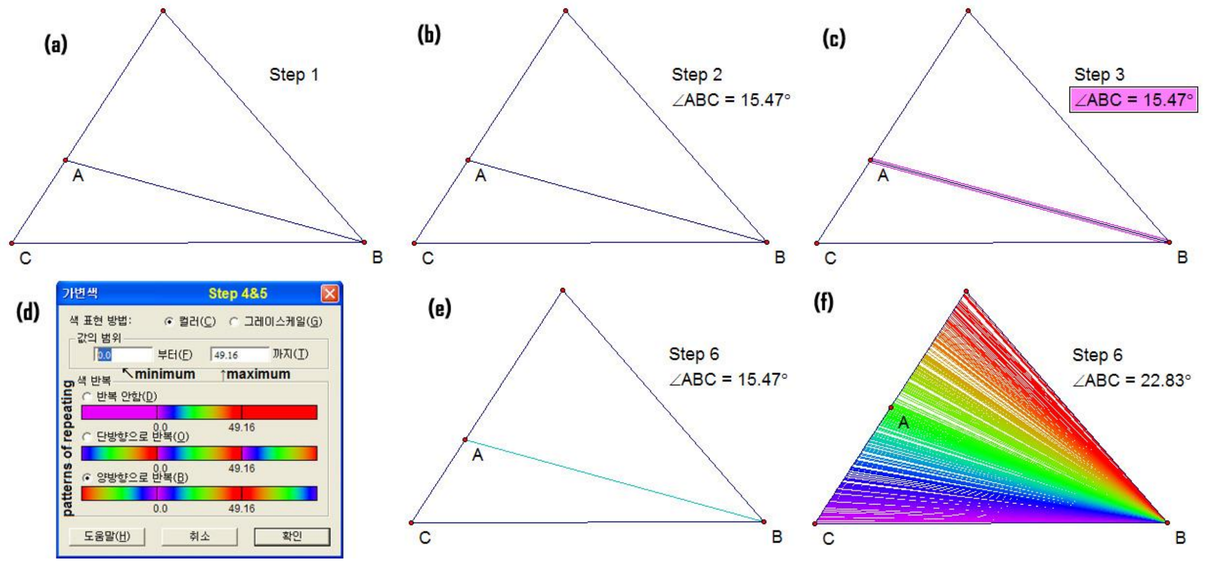
$$\frac{P_1Q_{1,2}}{Q_{1,2}P_2}\frac{P_2Q_{2,3}}{Q_{2,3}P_3}\cdots\frac{P_{n-1}Q_{n-1,n}}{Q_{n-1,n}P_n}\frac{P_nQ_{n,1}}{Q_{n,1}P_1}=1$$

4. Fermat Point

This chapter will discuss about advanced picturesque (or "colorful") method for quantity analysis that we will now say **quantity picturing**, using GSP. This method is for 2D geometric quantity analysis, or, to be specific, for max-min point analysis. Before we go on, we will first give formal definition of "quantity" that we will use so far.

Suppose we have subset D (usually constructed geometrically) of $\mathbb{R}^2$. We may consider a (continuous) map $f : \mathrm{D} \to \mathbb{R}$ that we will call (**continuous**) **quantity map** and the real number $f(\mathrm{P})$ will be called **quantity** at point P. Now to say how this concept is established in GSP, we just pick any point P from D, and take some measurements among these points (static points may be included during this procedure), and calculate these measurements to gain a quantity at P, and this generates quantity map from D also.

What we can do with GSP about the quantity is that they produce "coloring" figures – like points, lines or their segments, polygon areas, etc. – as the function of (given) quantity, or **controlling quantity**. Furthermore, if we trace a figure with varying color, it leaves its *former* color in their former positions (and therefore traces are static unless it's drawn once again). To implement it so, the detailed procedure is stated below [Figure 4-1].

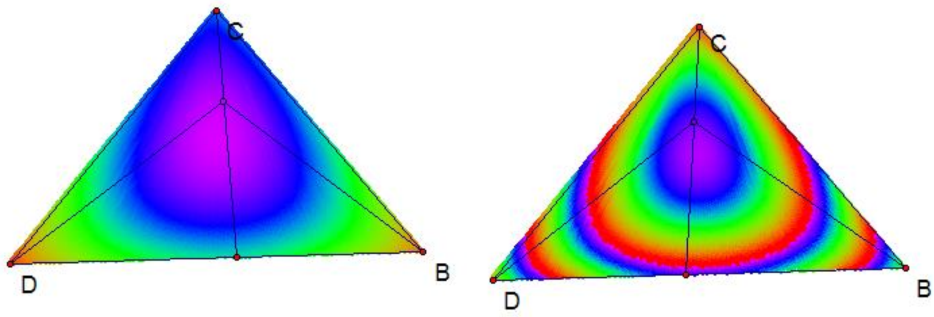


[Figure 4 - 1] Specific procedure of color conventional method

Step 1 – Draw a figure whatever you want to investigate. [(a)]
Step 2 – Give a quantity that you want to set as the controlling quantity. [(b)]
Step 3 – Select color-varying object and controlling quantity. [(c)]
Step 4 – Keeping the selection, go to View → Color → Varying color menu. [(d)]
Step 5 – Set the maximum, minimum value of repeating interval; be sure to notice what can be the maximum value and minimum value of controlling quantity in your figure. Set patterns of repeating if required. [(d)]
Step 6 – Now see how your varying-color object changes its aspect. If you need tracing it, you can turn on its tracing mode and see varying-color trace also. [(e),(f)]

Although example in [Figure 4-1] is giving traces of line segment, some specific cases that we usually deal with will trace a point P in domain D. Yet, this will be done by tracing the *curve trace* of point P in D, instead of point P itself – details will be depicted later. Whatever the method will be, this kind of traces throughout D will be called **quantity variation trace**. See [Figure 4-2] for example, which has 2 traces for the same quantity map f : D = Inferior of $\triangle BCD \to \mathbb{R}, P \mapsto BP + CP + DP$, and all of our varying-color traces will be assumed as a quantity variation traces from now on.

The most delicate part of this method is Step 5, setting repeating interval. The appropriate settings of this will enhance analysis of quantity variation trace of provided quantity map. If one sets repeating interval very similar to the same as range of quantity map, then one will see a **single color band**, or a single spectrum, from the point of minimum to the point of maximum. This kind of band is useful to see where the minimum or maximum value of quantity roughly exists, and to what direction is quantity rising; closely related with the concept of gradient as well. See the left side of [Figure 4-2] – one would find out that BP+CP+DP will have its minimum value somewhere in the middle of the triangle, and it increases as P moves to the boundary of triangle.



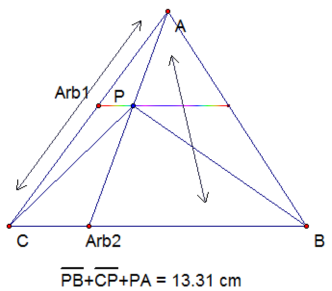
[Figure 4 - 2] Comparison between single color band & multiple color bands

For approximating quantities or figuring out more precisely where critical point exists, we should set repeating interval fairly small. If so, we find that variance of quantity may be shown as **multiple color bands**, that is, we may see multiple spectra of colors throughout the domain [Figure 4-2, right side]. The advantage of this is that we can resolute spectra more clearly. If continuity is assumed, then one can estimate which color band will represent which value of quantity, and moreover, one can figure out at which point will quantity map has its extreme values more clearly relative to single color band.

In the rest of this chapter, we will give an example of using quantity variation trace in geometric foundations. To be specific, we will apply it to search of Fermat's point, or, finding a point F that minimizes AF+BF+CF, for given ΔABC. There, all we have to do is examining a quantity variation trace, with quantity at P as AP+BP+CP and domain as ΔABC.

But there's one thing that we should be aware of; quantity picturing is bit hard to find critical point *exactly* from traces. This trace is no more than relying on person's sight, so our best information about the critical point from traces is that *the point exists possibly around somewhere*(=**critical site**), and this information is usually valid for continuous quantity only.

Therefore, using traces to find critical points should be no more than auxiliary tool of theorical construction of critical points of continuous quantities. Fortunately, for a planary point P, we have AP+BP+CP as continuous quantity, and for A, B, C satisfying certain conditions, we can geometrically construct critical point P inside ΔABC. So, we used traces to check validity of this geometrically constructed Fermat's point as an application, and at the end, we succeeded.



[Figure 4 - 3] Method for sweeping all points in triangle

There're two problems that we have to solve to do such works:

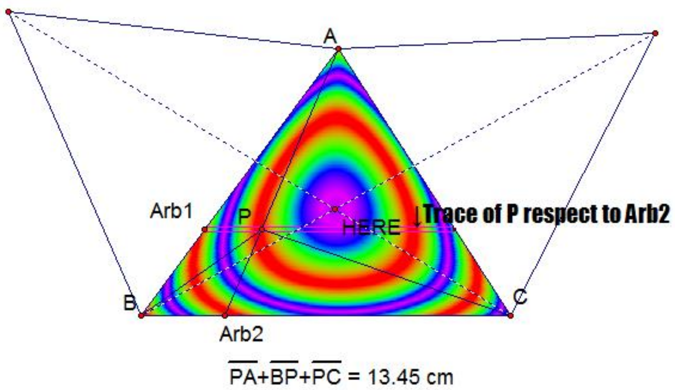
1. How we can check all points in ΔABC for tracing?

2. How we can solve problem 1 efficiently?

These 2 problems can be solved all at once using trace for coloring. The steps are described as following:

First, give 2 arbitrary point, say Arb1 and Arb2, on segment AB and BC, respectively. Second, give a parallel line by BC through Arb1, and link A with Arb2. The two lines designed will intersect at point P, and calculate AP+BP+CP and vary color of P respect to AP+BP+CP. One can ensure that if point Arb1 and Arb2 move totally randomly on their domain, point P will roam almost every point in triangle ABC.

However, AP+BP+CP will not be traced by point P itself; it'll be traced by the *curve* (or *linear*) *trace* of P respect to point Arb2 (or possibly Arb1, if you prefer so – we will note this as "tracing by *trace*.")[Figure 4-3]. That is, we're handling every point in triangle ABC by "tracing *trace*" of P. To do so, select point P and Arb2 with segment BC to generate P's *trace* by pt. Arb2. One can figure out easy that color is really varying inside the *trace* of point P, and this color(s) remain if we trace that *trace* (Italic *trace* refers to trace of P itself, and was to avoid ambiguity with other trace).



[Figure 4 - 4] Fermat's point & color band of quantity AP+BP+CP

(Presented with GSketchpad, [9])

Traced result of [Figure 4-3](P's *trace*) is shown as [Figure 4-4], with repeating interval (0, 0.5).

We indicated in [Figure 4-4] as "Here" for geometrically constructed Fermat's point. Note the area around point "Here" we can see that color bands are becoming bigger respect to their sizes, which is the important aspect of critical site. Therefore, we can see that [Figure 4-4] successfully gives validity of constructed Fermat's point. Moreover, although we won't discuss about the details, this method can be generalized for checking answers of general geometric optimization problems.

5. Conclusion

In summary, this paper contains generalizations of some triangle centers to tetrahedrons, and Menelaus' theorem, which introduced backward generalizations from 3D to 2D, and shows the process of finding Fermat point of a triangle, all done by some students who are used to usage of DGS. In some sense, our job was to make specific example of role of DGS as geometric knowledge producing by self-led way (for students).

Hence what we ultimately proposed in this paper is giving a specific example that shows how computer technique can help an extension of thought. For the example, we chose generalization of 2D-geometric property into 3D-geometric property. First, we chose geometry because it consists of studying our everyday figure, and, second, we chose generalization because it's simplest extensive thought upon the knowledge. The approach we used was dynamic modeling method using DGS. This way of study was very efficient and effective in research. We provided three example sub-studies using this method. In chapter 2, we provided an example of extending ideas directly from 2D to 3D. In addition, chapter 3 includes the capability of measuring in 3D, which can be impossible if we didn't use Cabri 3D. Also in chapter 4, we showed trial/mistake method of research and optimizing the tools in studying.

For decades, computer became one of the most important tools in human's daily lives. This means computers are interacting with people. As shown in this research, we showed it in this paper that this method is very efficient in studying mathematics. Through this study, we look forward to the future computer tools, both CAS and DGS, significantly interact with researchers and help their studies.

Acknowledgements

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Reference

[Kay01] David C. Kay, *College Geometry : A Discovery Approach 2nd Ed.*, Addision Wesley Longman, 2001.

Software Packages

[Cabri3D]
Cabri3D v.2, a product of CABRILOG, http://www.cabri.com/v2/pages/en/index.php
[GSketchpad]
Geometer's Sketchpad v.4, a product of Key Curriculum Press, http://www.dynamicgeometry.com/

Supplemental Electronic Materials

[1] *Kim, Dohyun, Cabri3D File (FiveCentersSample.cg3) containing the five centers of tetrahedrons, 2007*
[2] *Kim, Dohyun, Cabri3D File (Orthocenter_RightTetrahedron.cg3) containing the orthocenter of a right tetrahedron, 2007*
[3] *Kim, Dohyun, Cabri3D File (Orthocenter_FirstCon.cg3) containing the experiment on a tetrahedron moving the point upward and downward, 2007*
[4] *Kim, Dohyun, Cabri3D File (Orthocenter_CounterExample.cg3) containing the counter-example on conjectures for an orthocenter, 2007*
[5] *Kim, Dohyun, Cabri3D File (Orthocenter_undercondition.cg3) containing the final conditions*

for an orthocenter, 2007
[6] *Kim, Dohyun, Cabri3D File (Euler_Final.cg3) containing the generalization of Euler's Line on tetrahedrons, 2007*
[7] *Kim, Dohyun, Cabri3D File (Menelaus_Multipleloop.cg3) containing the first generalization on Menelaus' Theorem on on tetrahedrons: multi-loop, 2007*
[8] *Kim, Dohyun, Cabri3D File (Menelaus_3D.cg3) containing the circuit-concept generalization on tetrahedrons, 2007*
[9] *Jang, Seunguk, GSketchpad (FermatPtsEx.gsp) Containing the sample of quantity mapping: Fermat Point, 2007*